# Abundant Coherent Structures of the (2+1)-dimensional Broer-Kaup-Kupershmidt Equation

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By using of the Bäcklund transformation, which is related to the standard truncated Painlevé analysis, some types of significant exact soliton solutions of the (2+1)-dimensional Broer-Kaup-Kupershmidt equation are obtained. A special type of soliton solutions may be described by means of the variable coefficient heat conduction equation. Due to the entrance of infinitely many arbitrary functions in the general expressions of the soliton solution the solitons of the (2+1)-dimensional Broer-Kaup equation possess very abundant structures. By fixing the arbitrary functions appropriately, we may obtain some types of multiple straight line solitons, multiple curved line solitons, dromions, ring solitons and etc.

*Key words:* Bäcklund Transformation; (2+1)-dimensional Broer-Kaup-Kupershmidt Equation; Dromion Solution; Ring Solitons.

## 1. Introduction

In (1+1)-dimensions, the possible types of localized excitations like the bell type solitons, kinks, the breathers, compactons etc. have been deeply studied. However, the understanding on the possible localized excitations in (2+1) and (3+1)-dimensions is quite poor. Recently, it has been found that there exist much more abundant possible localized excitations in higher dimensions. The first type of a (2+1)-dimensional soliton, dromion, localized in all directions was found in [1]. The dromions in (2+1)-dimensions may be driven by several straight line solitons [2 - 4] and / or curved line solitons [5].

The (3+1)-dimensional dromions are also found for some types of Painlevé integrable models [6], conditionally integrable models [7] and even for nonintegrable models [8]. The dromions in (3+1)-dimensions may be driven by several plane solitons [7] and/or camber solitons [8, 6]. Very recently, one of the present authors (Lou) had found another type of localized solutions, ring solitons (which are nonzero only near a closed curve), both for some (2+1)-dimensional nonlinear physics models and for some types of (3+1)-dimensional models [6, 8 - 11].

The (1+1)-dimensional Broer-Kaup-Kupershmidt (BKK) system,

$$H_t - H_{xx} + 2(HH_x) + 2G_x = 0, \tag{1}$$

$$G_t + G_{xx} + 2(HG)_x = 0 \tag{2}$$

is one of the important integrable models which can be used to describe the propagation of long waves in shallow water [12]. The Hamiltonian and tri-Hamiltonian structure of the Model had been given by Kupershmidt [13]. Some types of the special solutions like the similarity solutions of (1) and (2) have been studied by many authors [15 - 16]. As for the KdV and Nonlinear Schrödinger equations, the BKK system can also be extended to some (2+1)-dimensional forms. One of the integrable extensions,

$$H_{ty} - H_{xxy} + 2(HH_x)_y + 2G_{xx} = 0, \tag{3}$$

$$G_t + G_{xx} + 2(HG)_x = 0 \tag{4}$$

may be obtained from the inner parameter dependent symmetry constraints of the KP (Kadomtsev-Petviashvilli) equation [17].

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In this paper we are interested to find some special solutions of the (2+1)-dimensional BKK system (3) and (4). Especially, we try to obtain some special types of localized solutions like dromions and the ring solitons.

The Painlevé analysis developed by Weiss, Tabor, and Carnevale (WTC) [18] is one of the most effective approaches to study the integrability of a nonlinear partial differential equation (PDE). Using the WTC approach to nonlinear PDEs, one can obtain some knowledge like the Painlevé property, Lax pair, bilinear form, Bäcklund transformation, etc. Especially the truncated Painlevé expansion and / or some types of its variant forms are widely used to find some kinds of exact solutions of nonlinear physics problems. In the next section, we use the truncated Painlevé expansion and the related Bäcklund transformation to look for a general formula of the solutions of the (2+1)-dimensional BKK system with many arbitrary functions. In section three we discuss some special types of localized solutions by fixing the arbitrary functions. The last section offers a short summary and discussion.

## 2. A General Type of Solutions with some Arbitrary Functions

Using the standard leading order analysis, the truncated Painlevé expansion of the (2+1)-dimensional BKK system may have the following form:

$$G = \frac{G_0}{f^2} + \frac{G_1}{f} + G_2, \quad H = \frac{H_0}{f} + H_1, \tag{5}$$

where $\{G_2,\ H_1\}$ is an arbitrary seed solution of the (2+1)-dimensional BKK system (3) and (4). That means, (5) denotes a Bäcklund transformation of the (2+1)-dimensional BKK system. For later convenience, we select the seed solution as

$$G_2 = 0, \quad H_1 = h(x,t), \tag{6}$$

where $h(x,t) \equiv h$ is an arbitrary function of $\{x,\ t\}$. Substituting (5) and (6) into (3) and (4) and vanishing the coefficients of the different powers of $f$, we get the following overdetermined system of eight equations to determine the four functions $H_0,\ G_0,\ G_1$ and $f$:

$$2H_{0y}h_x + 2hH_{0xy} - H_{0xxy} + 2G_{1xx} + H_{0ty} = 0, \tag{7}$$

$$\begin{aligned}&-2hf_xH_{0y} - 4G_{1x}f_x + 2H_0H_{0xy} - 2G_1f_{xx}\\ &- 2hf_yH_{0x} + H_{0xx}f_y + f_{xxy}H_0 + f_{xx}H_{0y}\\ &- 2hf_{xy}H_0 + 2f_xH_{0xy} - 2f_yH_0h_x + 2H_{0y}H_{0x}\\ &- f_yH_{0t} + 2f_{xy}H_{0x} - f_{ty}H_0 - f_tH_{0y} + 2G_{0xx} = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}&-2f_x^2H_{0y} - 2f_yf_{xx}H_0 + 4hf_yf_xH_0 - 4f_{xx}G_0\\ &- 4f_yH_0H_{0x} - 4f_xf_yH_{0x} - 4f_xH_0f_{xy} - 2f_{xy}H_0^2\\ &+ 4f_x^2G_1 - 8f_xG_{0x} - 4H_{0y}f_xH_0 + 2f_yf_tH_0 = 0,\end{aligned} \tag{9}$$

$$6f_yH_0^2f_x + 12f_x^2G_0 + 6f_yf_x^2H_0 = 0, \tag{10}$$

$$G_{1t} + 2h_xG_1 + 2hG_{1x} + G_{1xx} = 0, \tag{11}$$

$$\begin{aligned}&-2hG_1f_x - G_1f_{xx} + 2hG_{0x} + G_{0xx} - G_1f_t\\ &+ 2H_{0x}G_1 - 2G_{1x}f_x + 2h_xG_0 + 2H_0G_{1x} + G_{0t} = 0,\end{aligned} \tag{12}$$

$$\begin{aligned}&-4hf_xG_0 - 2f_{xx}G_0 + 2f_x^2G_1 + 2H_0G_{0x} - 2f_tG_0\\ &- 4f_xG_{0x} + 2H_{0x}G_0 - 4f_xH_0G_1 = 0,\end{aligned} \tag{13}$$

$$-6f_xH_0G_0 + 6f_x^2G_0 = 0. \tag{14}$$

After finishing some detailed calculations, we find that the overdetermined systems (7 - 14) can be solved by means of the following Ansatz

$$H_0 = f_x, \quad G_0 = -f_xf_y, \quad G_1 = f_{xy}, \tag{15}$$

while $f(x,y)$ satisfies the equation

$$f_t = -f_{xx} - 2hf_x. \tag{16}$$

Substituting (15) and (16) into (5), we get a quite general soliton solution of the (2+1)-dimensional BKK equation

$$H = (\ln f)_x + h, \quad G = (\ln f)_{xy}, \tag{17}$$

where $f$ is given by (16).

Because $h$ is an arbitrary function of $\{x,\ t\}$, a quite simple solution of (16) can be written as

$$f = Y_1 + Y_2F, \quad h = -\frac{1}{2}\left(\frac{F_t}{F_x} + \frac{F_{xx}}{F_x}\right), \tag{18}$$

where $F \equiv F(x,t)$ is an arbitrary function of $\{x,t\}$ and $Y_1 \equiv Y_1(y)$ and $Y_2 \equiv Y_2(y)$ are arbitrary function of $y$.

Substituting (18) into (17), we have

$$H = \frac{Y_2 F_x}{Y_1 + Y_2 F} - \frac{1}{2}\left(\frac{F_t}{F_x} + \frac{F_{xx}}{F_x}\right), \tag{19}$$

$$G = \frac{Y_{2y} F_x}{Y_1 + Y_2 F} - \frac{Y_2 F_x}{(Y_1 + Y_2 F)^2}(Y_{1y} + Y_{2y} F) \tag{20}$$

with arbitrary functions $Y_1, Y_2, F$ of their respective variables $y$ or $x, t$.

In [3], Radha and Lakshmanan gave out the dromion solutions which are driven by several straight line ghost solitons of the ANNV (asymmetric Niznnik-Novikov-Veselov) equation. The dromions decay exponentially in all directions and located at the intersection point of the straight line solitons. In [5], one of the present author (Lou) pointed out that the dromions of the ANNV may also be driven by curved line solitons. More recently, Lou [8] and Lou and Ruan [9] have found that there exist much more localized coherent structures like the ring solitons and many types of breathers and instantons for the NNV and ANNV equations.

From equations (19) and (20) we can see that because of the presence of the arbitrary functions $Y_1$, $Y_2$, and $F$ the (2+1)-dimensional BKK system has an abundance of localized coherent structures created by selecting those arbitrary functions appropriately.

## 3. Special Examples

### 3.1. Multi-straight Line Solitons: "Multi-solitoff Solutions"

If we take $Y_1 = 1$ and $F$ and $Y_2$ as linear combinations of the exponentials of $k_i x + \omega_i t + x_{i0} \equiv \xi_i$ and $K_i y + y_{i0} \equiv \eta_i$respectively, i.e.

$$F = \sum_{i=1}^{N} \exp(\xi_i), \quad Y_2 = \sum_{i=1}^{M} \exp(\eta_i), \tag{21}$$

where $x_{0i}$, $y_{0i}$, $k_i$, $\omega_i$, and $K_i$ are arbitrary constants and $M$ and $N$ are arbitrary positive integers, we obtain the first type of the multiple straight line solitons. Figure 1 shows the special structure of a two straight line soliton for the field $G$ with

$$F = \frac{1}{5}(\exp(x + \omega_1 t) + \exp(2x + \omega_2 t - 1)),$$

$$Y_1 = 1, \; Y_2 = \exp(y) + \exp(3y - 1) \tag{22}$$

at $t = 0$. From the Fig. 1, one can see that the two straight line soliton is localized on two half straight lines with "V" shape and other two half lines had vanished because of the interaction. This type of half straight line solitons is called solitoff solution in the literature. Actually, this type of solutions has been found for many other (2+1)-dimensional integrable models, say, the KP equation. The known "Y" shaped three straight line soliton of the KP equation is just the three solitoff solution. This type of solitoff solution may also be called as resonant straight line solitons. For a multiple straight line soliton solution it is necessary to include some coupling terms among different ones to guarantee that straight line solitons are *full* straight line solitons. For instance, for a KdV type equation, say the KP equation, a two full straight line soliton solution should have the form $(a \ln f)_{xx}$ with $f = 1 + \exp(k_1 x + l_1 y + \omega_1 t) + \exp(k_2 x + l_2 y + \omega_2 t) + a_{12} \exp(k_1 x + l_1 y + \omega_1 t) \exp(k_2 x + l_2 y + \omega_2 t)$, where the final term is a coupling term. If the coupling term is canceled ($a_{12} = 0$), then the resonance appears such that half of the straight line solitons disappear. mathematically, the solution without the coupling term is still valid in both directions, however, when $\exp(k_i x + l_i y + \omega_i t) \to \infty$, the denominator will be $\infty^2$, while the numerator is only $\infty^1$ when $a_{12} = 0$. That is why half of the straight line solitons will disappear.

### 3.2. Dromion Solutions Induced by Multiple Straight Line and Curved Line Solitons

For convenience of later discussion we take the transformation

$$F \equiv \exp(2p), \; Y_2 \equiv \exp(2q), \tag{23}$$

where $p \equiv p(x, t)$ and $q \equiv q(y)$ are still arbitrary functions because of the arbitrariness of $F$ and $Y_2$. Substituting (23) into (20) and taking $Y_1 = 1$, (20) becomes

$$G = \frac{1}{4} q_y p_x \operatorname{sech}^2(p + q). \tag{24}$$

From (24), one can easily see that many kinds of soliton solutions may be driven by two sets of straight line solitons and one set of curved line solitons. The first set straight line solitons are parallel to the $x$ axis

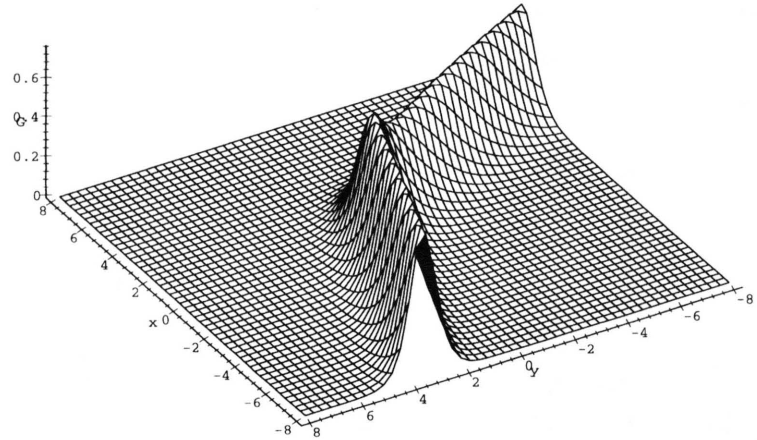



Fig. 1. Plot of the two solitoff solution for $G$ with the condition (22) at $t = 0$.

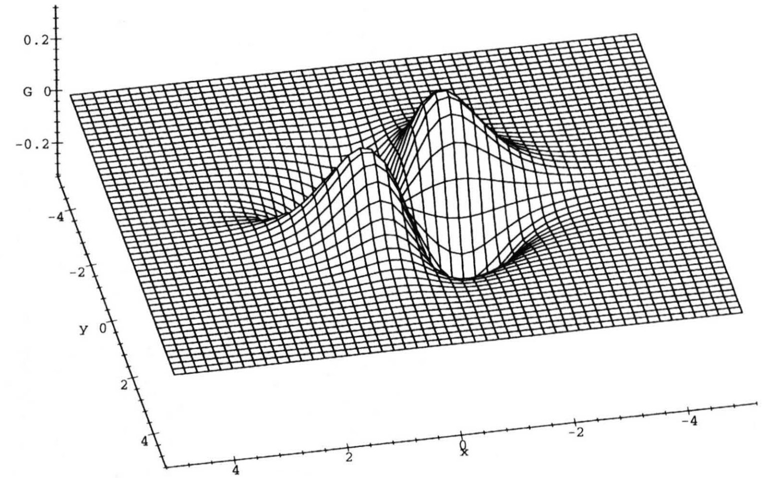



Fig. 2. A special type of Dromion solution for the field $G$ driven by two straight line solitons (26) at time $t = 0$.

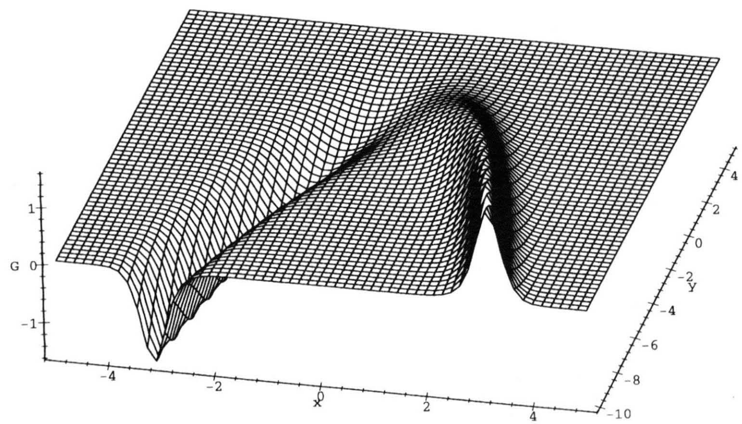



Fig. 3. A plot of a special curved line soliton solution with condition (27) for the field $G$ at time $t = 0$.

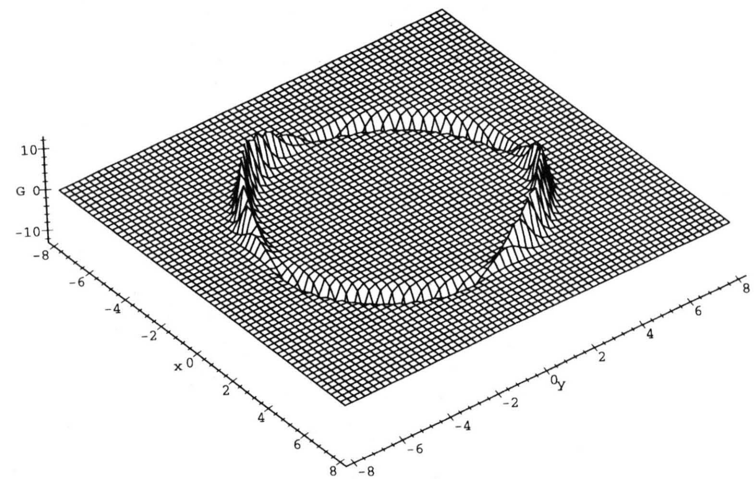



Fig. 4. A special ring soliton solution related to (28) for the field $G$ at time $t = 0$.

and determined by $q_y$. We may select $q_y$ as, say,

$$q_y = \sum_{i=1}^{N} Q_i(y - y_i),$$

where $Q_i(y - y_i)$ may be some localized solitons located at $y = y_i$. The second set straight line solitons are parallel to the $y$ axis and determined by $p_x$. For instance, one can select $p_x$ as an arbitrary multiple soliton solution for arbitrary (1+1)-dimensional integrable models, or one can also select $p_x$ as

$$p_x = \sum_{i=1}^{M} P_i(x - x_i(t), t),$$

with $P_i(x - x_i(t), t)$ being some type of localized straight line soliton located at $x = x_i(t)$. The set of curved line solitons is given by the factor $\mathrm{sech}^2(p+q)$. They are located at

$$p + q = \min |p + q|. \tag{25}$$

Figure 2 is a plot of a special dromion solution for the field $G$ at time $t = 0$ when $p$ and $q$ are chosen as

$$p = 2\ln(\mathrm{sech}(x + \omega t)),\ q = 2\ln(\mathrm{sech} y). \tag{26}$$

Figure 3 shows a special curved line solitons at $t = 0$ for the field $G$ with

$$p = (x + \omega t)^2,\ q = y. \tag{27}$$

### 3.3. Saddle-type Ring Soliton Solutions

In higher dimensions, in addition to the point-like (with one or more peak(s)) soliton solutions, there may be other types of localized coherent solutions. Especially, some different types of ring (a closed curve) soliton solutions have been found in (2+1)-dimensions, say, the plateau-type, basin-type and bowl-type ring solitons [19] for the (2+1)-dimensional sine-Gordon equation [20] and the saddle-type of ring soliton solutions for the NNV equation and ANNV equation [9 - 11]. Some special types of ring soliton solutions in (3+1)-dimensions have also been found [8, 6].

For the (2+1)-dimensional BKK system (1) and (2), some special types of ring solitons can also be found. For instance, if we choose $p$ and $q$ such that (25) describes a closed curve, then (24) may be a ring soliton solution. Figure 4 shows the structure of a special saddle type ring soliton solution for the field $G$ at time $t = 0$ for

$$p = (x + \omega t)^2 - 25,\ \ q = y^2. \tag{28}$$

### 3.4. Breathers and Instantons

Because $p$ is an arbitrary function of $\{x, t\}$, if some types of periodic functions and quickly decaying functions of time $t$ are included in $p$ then the soliton solution (24) will become some types of breathers and instantons. For instance, if all the choices of $p$ (or $F$) in (22), (23), (26), (27) and (28) are multiplied by the periodic (or decaying) function $(2 + \cos t)$ (or $\mathrm{sech} t$), then all the solutions shown in Figs. 1 - 4 become breathers (or instantons).

Figure 5 shows another type of saddle type of ellipsoidal ring breather solutions for the field $G$ with

$$p = \left(\frac{x}{2 + \cos t}\right)^2 - 25\left(1 - \frac{1}{2}\cos t\right),\ q = y^2. \tag{29}$$

From Fig. 5, we see that the ellipsoidal breather breathes not only in its amplitude (from about 2 to 10) but also in its shape like the axes of the ellipse.

### 3.5. Other Types of Exact Solutions

Actually, because (16) is only linear in $f$, we may obtain many other types of exact solutions. For instance, we can take

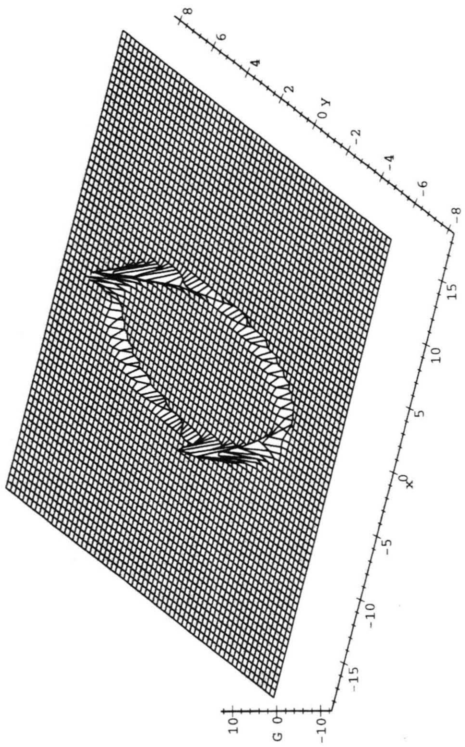
$$f = 1 + \sum_{i=1}^{N} Y_i F_i,\ h = -\frac{1}{2}\left(\frac{F_{1t}}{F_{1x}} + \frac{F_{1xx}}{F_{1x}}\right), \tag{30}$$

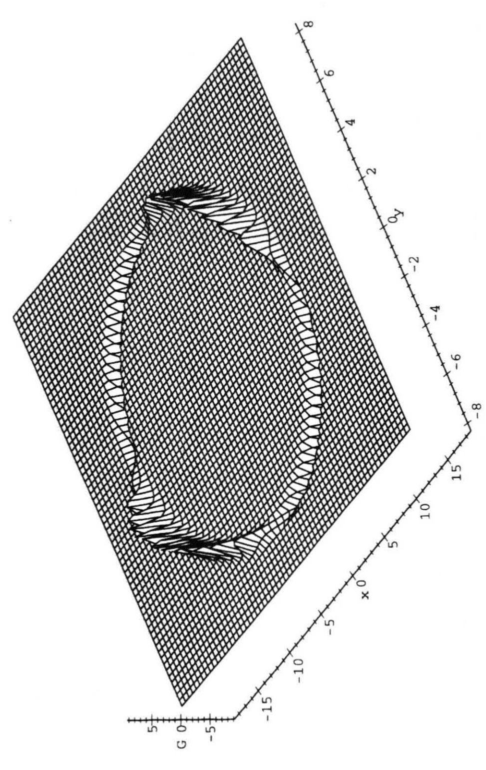
with $Y_i$ being arbitrary functions of $y$, $F_1$ being arbitrary functions of $\{x, t\}$ while $F_i, i = 2, 3, ..., N$ are $N - 1$ independent solutions of

$$(F_{ixx} + F_{it})F_{1x} - (F_{1xx} + F_{1t})F_{ix} = 0, \tag{31}$$

$$i = 2, 3, ..., N.$$

The solution (18) discussed above is just the simplest special case of (30).



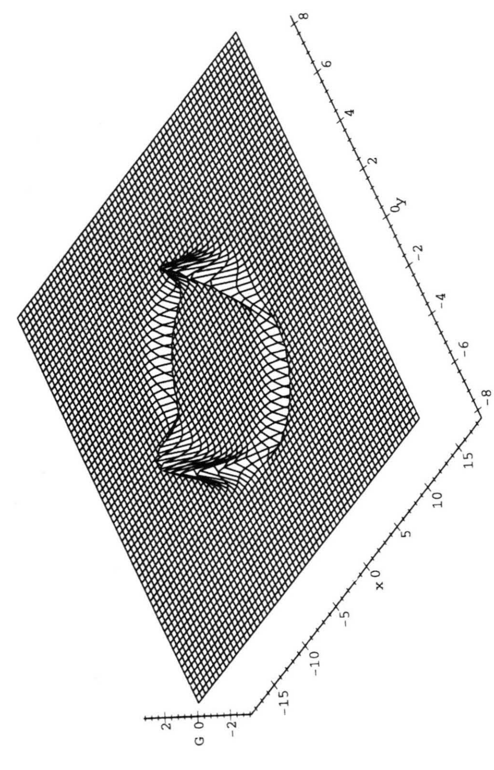




Fig. 5. A ring type breather with (29) at time $t = 0$, $2\pi$ (*top, left*), $t = \pi/2$ (*top, right*), and $t = \pi$ (*left*).

If we take $h$ as an arbitrary function of $t$ only, then a quite general solution of (16) can be taken as

$$f = \frac{1}{2\sqrt{\pi(t_0 - t)}} \int_{\infty}^{\infty} \phi(\xi, y) \tag{32}$$

$$\cdot \exp \frac{(\xi - x - 2\int_0^{t_0 - t} h(\tau)\,\mathrm{d}\tau - Y(y))^2}{4(t_0 - t)} \mathrm{d}\xi,$$

where $t_0$ is a constant, $Y(y)$ is an arbitrary function of $y$, and $\phi(\xi, y)$ is an arbitrary function of $\xi$ and $y$.

The corresponding solutions of the BKK system can be obtained by substituting (30) and/or (31) into (17).

## 4. Summary and Discussions

In this paper, using the standard Painlevé analysis and the related Bäcklund transformation to solve the (2+1)-dimensional BKK system, many interesting localized coherent solutions are obtained. The richness of the solutions comes from the presence of some arbitrary functions. By selecting the arbitrary functions appropriately, some types of solitoff solutions, curved lines solitons, dromions and saddle type ring solitons are listed and plotted also.

Because some types of time dependent solutions can be included in the solutions quite freely, all the localized solutions of the BKK system may temporally evolve in many ways. Especially, the ring type breathers may breath in some different ways, say, a ring breather may change its amplitude and shape (axes of the ellipse) periodically.

Such a richness of localized coherent structures has been found not only for the (2+1)-dimensional BKK system described here, but also for some other important (2+1)-dimensional nonlinear integrable models like the sG, NNV, ANNV and DS equations and even for some (3+1)-dimensional models [6, 8]. We believe that this is a general phenomenon in high dimensions because some types of arbitrary functions may appear in the partial integrations of high dimensional partial differential equations. Because of the wide applications of the soliton theory in real physics, more about the properties of the high dimensional localized solutions, like the ring soliton solutions, is worth of further study.

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